Generalized Binomial Thm : First for $\alpha \in (0,1)$.

Let $\alpha \in (0,1)$, for $x \in (-1,1)$, we have

$$(1-x)^\alpha = 1 - \alpha x - \frac{\alpha(1-\alpha)}{2!}x^2 - \cdots - \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}x^n - \cdots$$

Some Attempt : (Using Taylor's thm of Lagrange's form).

$$[(1-x)^\alpha]^{(k)} = -\alpha(1-\alpha)\cdots(k-1-\alpha)\frac{1}{(1-x)^{k-\alpha}} \quad \text{for } k \geq 2$$

$$[(1-x)^\alpha]^{(1)} = -\alpha\frac{1}{(1-x)^{1-\alpha}}$$

It remains to check the error term $E_n(x)$

$$E_n(x) = -\frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}\frac{1}{(1-\xi_x)^{n-\alpha}}x^n \quad , \xi_x \text{ strictly bet' } 0 \text{ and } x$$

For $-1<x<0$, then $1-\xi_x > 1$ and for $n>1$,

$$|E_n(x)| = \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}\cdot\frac{1}{(1-\xi_x)^{n-\alpha}}|x|^n = \alpha\cdot\frac{1-\alpha}{1}\cdot\frac{2-\alpha}{2}\cdots\frac{n-1-\alpha}{n-1}\cdot\frac{1}{n}\frac{1}{(1-\xi_x)^{n-\alpha}}|x|^n$$

$$\leq \frac{1}{n} \to 0 \text{ as } n\to\infty$$

For $0<x<\frac{1}{2}$, then $1-\xi_x > \frac{1}{2}$ and for $n>1$,

$$|E_n(x)| \leq \frac{1}{n}\cdot 2^{n-\alpha}\cdot\left(\frac{1}{2}\right)^n = \frac{1}{2^\alpha n} \to 0 \text{ as } n\to\infty.$$

For $x=\frac{2}{3}$, then $(1-\xi_x) > \frac{1}{3}$, for $n>1$,

$$|E_n(x)| = \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}\cdot\frac{1}{(1-\xi_x)^{n-\alpha}}\left(\frac{2}{3}\right)^n$$

$$\leq \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}\cdot 3^{n-\alpha}\left(\frac{2}{3}\right)^n = \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!}2^n\cdot\frac{1}{3^\alpha}$$

Try $\alpha=\frac{1}{2}$, we have
$$\frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-3}{2})\cdot 2^n}{n!}\frac{1}{\sqrt{3}} = \frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{n!}\cdot\frac{1}{\sqrt{3}}$$

which does not converge to zero.

Some Reason : Relation of ξ_x and x is not known much.

Instead, we use Taylor's thm with integral term, the proof is from integration by parts. Let $f:[0,1]\to\mathbb{R}$, for $0<x<1$,

$$f(x) = \int_0^x f'(t)\,dt + f(0) \quad \text{by Fundamental theorem of Calculus}$$

$$= f(0) - \int_0^x f'(t)\,d(x-t) = f(0) - f'(t)(x-t)\Big|_{t=0}^{x} + \int_0^x f''(t)(x-t)\,dt$$

$$= f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt \ldots$$

Assuming $f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\,dt \qquad n \ge 2$

$$= f(0) + f'(0)x + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} - \int_0^x \frac{f^{(n)}(t)}{n!}\,d\left((x-t)^n\right)$$

$$= f(0) + f'(0)x + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n + \int_0^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt \qquad \#$$

The error term is $\int_0^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$.

Putting $f(x) = (1-x)^\alpha$ .. $\alpha \in (0,1)$, fix $x \in (0,1)$

$$E_n(x) = \int_0^x \frac{-\alpha(1-\alpha)\cdots(n-\alpha)}{(n\ )!}\left(\frac{1}{1-t}\right)^{n+1-\alpha}\cdot(x-t)^n\,dt$$

$$|E_n(x)| \le \alpha\int_0^x \left(\frac{x-t}{1-t}\right)^n\cdot\frac{1}{(1-t)^{1-\alpha}}\,dt$$

$$\le \frac{\alpha}{(1-x)^{1-\alpha}}\int_0^x \left(\frac{x-t}{1-t}\right)^n\,dt$$

Note that $0 \le \left(\frac{x-t}{1-t}\right) < 1$ for $t \in (0,x]$

$$\frac{d}{dt}\left(\frac{x-t}{1-t}\right) = \frac{-(1-t)-(x-t)(-1)}{(1-t)^2} = \frac{x-1}{(1-t)^2} < 0$$

$$\therefore \left(\frac{x-t}{1-t}\right) \le \frac{x-0}{1-0} = x \quad \forall\ t \in (0,x]$$

$$\therefore |E_n(x)| \le \frac{\alpha}{(1-x)^{1-\alpha}}\int_0^x x^n\,dt = \frac{\alpha}{(1-x)^{1-\alpha}}x^{n+1} \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$ The special case of Generalized Binomial Thm is proved.

For the General one, use radius of convergence and differentiating the series termwisely or integrating the series termwisely.

You can also do that by $(1-x)^\alpha = \exp(\alpha\log(1-x))$

$$= \exp\left(-\alpha\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)$$

$$= 1 + \frac{\left(-\alpha\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)}{1!} + \frac{\left(-\alpha\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)^2}{2!} + \ldots$$

$$= 1 + a_1x + a_2x^2 + \ldots$$

then $(1-x)^\alpha$ can be written as power series, however, the term a_k is determined.